

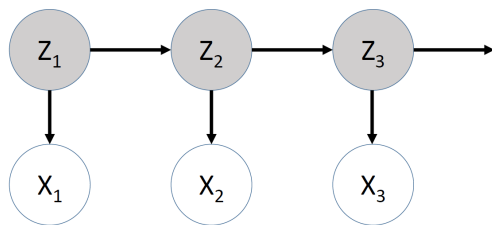
Algorithmic Trading: Hidden Markov Models

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Hidden Markov Models

HMM



- ▶ $Z = \{Z_t : t = 1, \dots, N\}$ is a K -state Markov chain, transition matrix A , modelling the **unobserved** or **latent** states
- ▶ $X = \{X_t : t = 1, \dots, N\}$ are the **observed data** with **emission probability**

$$\psi_i(x) := \mathbb{P}(X_t = x \mid Z_t = i)$$

[Note: it is independent of time]

- The **completed-data log-likelihood** for observations $\mathbf{x} = \{x_1, \dots, x_n\}$ is

$$\begin{aligned}\bar{\ell}(\Theta; \mathbf{Z}) = & \sum_{i=1}^K (\log \pi_i) \mathbb{1}_{\{Z_1=i\}} \\ & + \sum_{t=1}^n \sum_{i=1}^K (\log \psi_i(x_t)) \mathbb{1}_{\{Z_t=i\}} \\ & + \sum_{t=1}^{n-1} \sum_{i,j=1}^K (\log A_{ij}) \mathbb{1}_{\{Z_t=i, Z_{t+1}=j\}}\end{aligned}$$

HMM

- So that from the **e-step**

$$\begin{aligned}\bar{\ell}(\Theta) = & \sum_{i=1}^K (\log \pi_i) \gamma_1^i \\ & + \sum_{t=1}^n \sum_{i=1}^K (\log \psi_i(x_m)) \gamma_t^i \\ & + \sum_{t=1}^{n-1} \sum_{i,j=1}^K (\log A_{ij}) \xi_t^{ij}\end{aligned}$$

- where

smoother $\gamma_t^i := \mathbb{P}(Z_t = i \mid X_{1:T} = x_{1:T})$

two-slice marginal $\xi_t^{ij} = \mathbb{P}(Z_t = i, Z_{t+1} = j \mid X_{1:T} = x_{1:T})$

- From the **m-step** we have the update rule

$$\pi_i = \frac{\gamma_1^i}{\sum_i \gamma_1^i}$$

$$\psi_i(x) = \frac{\sum_t \gamma_t^i \mathbb{1}_{x_t=x}}{\sum_t \gamma_t^i}$$

$$A_{ij} = \frac{\sum_t \xi_t^{i,j}}{\sum_t \sum_j \xi_t^{i,j}}$$

The following quantities are useful for estimating the model:

forward filter $\alpha_t^i := \mathbb{P}(Z_t = i \mid X_{1:t} = x_{1:t})$

$$\eta_t := \mathbb{P}(X_t = x_t \mid X_{1:t-1} = x_{1:t-1})$$

backwards filter $\beta_t^i := \frac{\mathbb{P}(X_{t+1:T} = x_{t+1:T} \mid Z_t = i)}{\mathbb{P}(X_{t+1:T} = x_{t+1:T} \mid X_{1:t} = x_{1:t})}$

HMM

The forward recursion:

$$\alpha_1^i = \psi_i(x_1) \pi^i$$

initialize

$$\tilde{\alpha}_t^i = \psi_i(x_t) \sum_{j=1}^K A_{ij}^T \alpha_{t-1}^j$$

update

$$\eta_t = \sum_{i=1}^K \tilde{\alpha}_t^i$$

normalization factor

$$\alpha_t^i := \frac{\tilde{\alpha}_t^i}{\eta_t}$$

normalize

The backward recursion:

$$\beta_T^i = 1$$

initialize

$$\beta_t^i = \frac{1}{\eta_{t+1}} \sum_{j=1}^K A_{ij} \beta_{t+1}^j \psi_j(x_{t+1})$$

update

The smoother can be obtained as follows

$$\gamma_t^i = \frac{1}{Z} \alpha_t^i \beta_t^i$$

$$Z = \sum_{j=1}^K \alpha_t^j \beta_t^j$$

normalization

The smoothed two-slice marginal can be obtained as follows

$$\xi_t^{ij} = \frac{1}{Z} \alpha_t^i A_{ij} \psi_j(x_{t+1}) \beta_{t+1}^j$$

$$Z = \sum_{i,j=1}^K \alpha_t^i A_{ij} \psi_j(x_{t+1}) \beta_{t+1}^j \quad \text{normalization}$$

HMM

Using **signed price changes** each second

| Regime | INTC | | | | | |
|--------|-------|------|------|--------|------|------|
| | π | A | | ψ | | |
| | | | | | | |
| 1 | 0.01 | 0.99 | 0.01 | 0.02 | 0.95 | 0.02 |
| 2 | 0.99 | 0.01 | 0.99 | 0.08 | 0.85 | 0.07 |

| | SMH | | | | | |
|---|-------|------|------|--------|------|------|
| | π | A | | ψ | | |
| | | | | | | |
| 1 | 0.00 | 0.97 | 0.03 | 0.04 | 0.91 | 0.04 |
| 2 | 1.00 | 0.04 | 0.96 | 0.18 | 0.65 | 0.18 |

| | NTAP | | | | | |
|---|-------|------|------|--------|------|------|
| | π | A | | ψ | | |
| | | | | | | |
| 1 | 0.04 | 0.96 | 0.04 | 0.01 | 0.97 | 0.02 |
| 2 | 0.96 | 0.06 | 0.94 | 0.15 | 0.68 | 0.16 |

HMM

Using **volume imbalance** (sell,neutral,buy) each second

| Regime | INTC | | | | | |
|--------|-------|------|------|--------|------|------|
| | π | A | | ψ | | |
| 1 | 0.22 | 0.92 | 0.08 | 0.99 | 0.01 | 0.00 |
| 2 | 0.78 | 0.04 | 0.96 | 0.00 | 0.62 | 0.38 |

| | SMH | | | | | |
|---|-------|------|------|--------|------|------|
| | π | A | | ψ | | |
| 1 | 0.95 | 0.95 | 0.05 | 0.53 | 0.02 | 0.44 |
| 2 | 0.05 | 0.14 | 0.86 | 0.01 | 0.99 | 0.01 |

| | NTAP | | | | | |
|---|-------|------|------|--------|------|------|
| | π | A | | ψ | | |
| 1 | 0.73 | 0.98 | 0.02 | 0.00 | 0.61 | 0.39 |
| 2 | 0.27 | 0.09 | 0.91 | 0.99 | 0.01 | 0.00 |

HMM

Using **buy, sell order-flow** (one,one,both) each second

| Regime | INTC | | | | | | |
|--------|-------|------|------|------|--------|------|------|
| | π | A | | | ψ | | |
| 1 | 0.49 | 0.91 | 0.00 | 0.09 | 0.74 | 0.23 | 0.03 |
| 2 | 0.00 | 0.01 | 0.99 | 0.00 | 0.46 | 0.35 | 0.19 |
| 3 | 0.51 | 0.02 | 0.00 | 0.98 | 0.95 | 0.04 | 0.01 |

| Regime | SMH | | | | | | |
|--------|-------|------|------|------|--------|------|------|
| | π | A | | | ψ | | |
| 1 | 0.38 | 1.00 | 0.00 | 0.00 | 0.99 | 0.01 | 0.00 |
| 2 | 0.56 | 0.00 | 1.00 | 0.00 | 0.97 | 0.03 | 0.00 |
| 3 | 0.05 | 0.16 | 0.00 | 0.84 | 0.88 | 0.11 | 0.01 |