Algorithmic Trading : Optimal Execution

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many thanks to

Álvaro Cartea, Oxford

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Optimal Execution

- How do we trade **in** or **out** of a large position?
- How trades affect prices?
  - **temporary** price impact
  - **permanent** price impact
- What is the level of urgency to execute the large position?

Model Setup

- $\nu = (\nu_t)_{\{0 \le t \le T\}}$, speed at which the agent is liquidating or acquiring shares (**MOs only**)
- $Q^\nu = (Q_t^\nu)_{\{0 \le t \le T\}}$, inventory
- $S^\nu = (S_t^\nu)_{\{0 \le t \le T\}}$, midprice
- $\hat{S}^\nu = (\hat{S}_t^\nu)_{\{0 \le t \le T\}}$, execution price (by walking the LOB)
- $X^\nu = (X_t^\nu)_{\{0 \le t \le T\}}$, agent's cash process

Model Setup

- Inventory

$$dQ_t^\nu = \pm\nu_t\, dt\,, \qquad\qquad Q_0^\nu = q\,,$$

- Midprice

$$dS_t^\nu = \pm \mathbf{g}(\boldsymbol{\nu}_\mathbf{t})\, dt + \sigma\, dW_t\,, \qquad S_0^\nu = S\,,$$

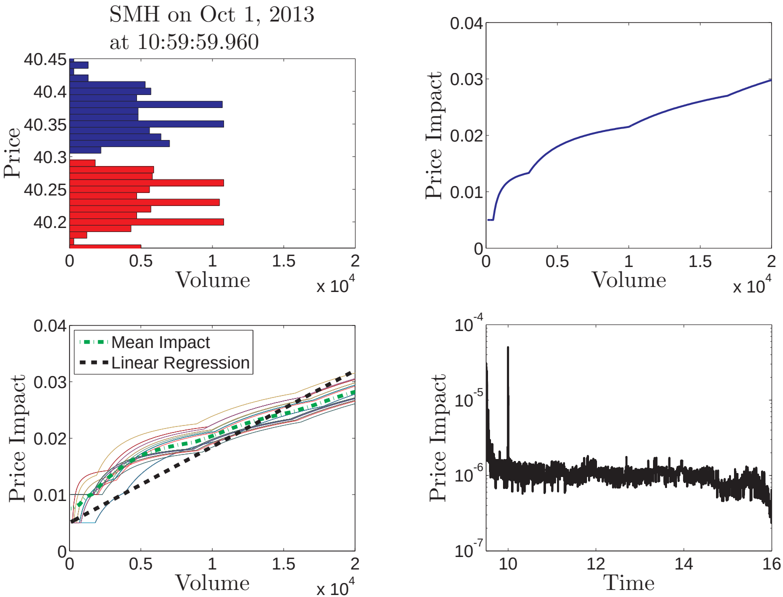
  - $W = (W_t)_{\{0\le t\le T\}}$, standard Brownian motion,
  - $g : \mathbb{R}_+ \to \mathbb{R}_+$, **permanent price impact**.

- Execution price

$$\hat{S}_t^\nu = S_t^\nu \pm \left(\tfrac{1}{2}\Delta + \mathbf{f}(\boldsymbol{\nu}_\mathbf{t})\right)\,, \qquad \hat{S}_0^\nu = \hat{S}\,,$$

  - $f : \mathbb{R}_+ \to \mathbb{R}_+$, **temporary price impact**
  - $\Delta \ge 0$, bid-ask spread

SMH on Oct 1, 2013



Optimal Liquidation with Temporary Impact

Liquidation

- Value function

$$H(t,S,q) = \sup_{\nu\in\mathcal{A}} \mathbb{E}_{t,S,q}\left[\int_t^T (S_u - \mathbf{a}\,\nu_{\mathbf{u}})\,\nu_u\,du\right].$$

- $f(\nu) = \mathbf{a}\,\nu$, and $g(\nu) = 0$.

- The DPP suggests that H

$$\partial_t H + \tfrac{1}{2}\sigma^2\,\partial_{SS}H + \sup_{\nu}\left\{(S - \mathbf{a}\,\nu)\,\nu - \nu\,\partial_q H\right\} = 0\,.$$

Agent must liquidate all inventory by time T:

$$H(T,S,q) \xrightarrow{t\to T} -\infty, \quad \text{for} \quad q>0\,,$$

$$H(T,S,0) \xrightarrow{t\to T} 0\,.$$

Liquidation

Optimal liquidation:

$$\nu^* = \frac{1}{2\,\mathbf{a}}\left(S - \partial_q H\right) .$$

So value function satisfies

$$\partial_t H + \tfrac{1}{2}\,\sigma^2\,\partial_{SS} H + \frac{1}{4\,\mathbf{a}}\left(S - \partial_q H\right)^2 = 0\,.$$

Propose ansatz

$$H(t,q) = \underbrace{q\,S}_{\text{book value at mid}} + \underbrace{h(t,q)}_{\text{value from optimal trading}} ,$$

thus

$$\partial_t h + \frac{1}{4\,\mathbf{a}}\left(\partial_q h\right)^2 = 0\,.$$

- Make ansatz $h(t,q) = q^2\, h_2(t)$:

$$\partial_t h_2 + \tfrac{1}{\mathbf{a}}\, h_2^2 = 0\,.$$

- Integrate between t and T

$$h_2(t) = \left(\frac{1}{h_2(T)} - \frac{1}{\mathbf{a}}(T-t)\right)^{-1}\,.$$

- Note that $h_2(T) \to -\infty$ as $t \to T$, hence

$$\boxed{\nu_t^* = \frac{1}{T-t}\, Q_t^{\nu^*}\,,}$$

which is known as **TWAP**.

Optimal Acquisition with Temporary Impact

Acquisition

- Objective acquire $\mathfrak{N}$ shares by time T.
- Execution costs

$$\hat{S}^\nu_t = S^\nu_t + \left(\tfrac{1}{2}\Delta + \mathbf{a}\,\boldsymbol{\nu}_\mathbf{t}\right)\,.$$

- Expected costs from strategy ν_t

$$EC^\nu = \mathbb{E}\Big[\underbrace{\int_t^T \hat{S}_t\,\nu_u\,du}_{\text{Terminal Cash}} + \underbrace{(\mathfrak{N} - Q^\nu_T)\,S_T}_{\text{Terminal execution at mid}} + \underbrace{\alpha\,(\mathfrak{N} - Q^\nu_T)^2}_{\text{Terminal Penalty}}\Big]\,.$$

- Let $Y^\nu = (Y^\nu_t)_{0\le t\le T}$, shares remaining to be purchased:

$$Y^\nu_t = \mathfrak{N} - Q^\nu_t\,, \qquad \text{so that} \qquad dY^\nu_t = -\nu_t\,dt\,.$$

- Value function

$$H(t,S,y) = \inf_{\nu\in\mathcal{A}} \mathbb{E}_{t,S,y}\left[\int_t^T \hat{S}_u\,\nu_u\,du + Y^\nu_T\,S_T + \alpha\,\left(Y^\nu_T\right)^2\right]\,.$$

- H satisfies

$$0 = \partial_t H + \tfrac{1}{2}\sigma^2 \partial_{SS} H + \inf_{\nu} \left\{ (S + \mathbf{a}\,\nu)\,\nu - \nu \partial_y H \right\}\ ,$$

with $H(T, S, y) = y\,S + \alpha\, y^2$.

- Optimal acquisition speed in feedback form

$$\boxed{\nu^* = \tfrac{1}{2\,\mathbf{a}}\left(\partial_y H - S\right)\ ,}$$

- so

$$\partial_t H + \tfrac{1}{2}\sigma^2 \partial_{SS} H - \tfrac{1}{4\,\mathbf{a}}\left(\partial_y H - S\right)^2 = 0\,.$$

- Make ansatz

$$H(t, S, y) = y\,S + h_0(t) + h_1(t)\,y + h_2(t)\,y^2\,,$$

with terminal conditions

$$h_2(T) = \alpha \quad \text{and} \quad h_1(T) = h_0(T) = 0\,.$$

Upon substituting the ansatz into PDE above:

$$0 = \left\{\partial_t h_2 - \tfrac{1}{\mathbf{a}}\, h_2^2\right\}\, y^2 + \left\{\partial_t h_1 - \tfrac{1}{2\,\mathbf{a}}\, h_2\, h_1\right\}\, y + \left\{\partial_t h_0 - \tfrac{1}{4\,\mathbf{a}}\, h_1^2\right\}\ .$$

- Due to $h_1(T) = 0$, we obtain $h_1(t) = 0$.
- Similarly, since $h_0(T) = 0$ and $h_1(t) = 0$ we obtain $h_0(t) = 0$.
- Finally, since $h_2(T) = \boldsymbol{\alpha}$

$$h_2(t) = \left(\tfrac{1}{\mathbf{a}}\,(T - t) + \tfrac{1}{\boldsymbol{\alpha}}\right)^{-1}\ .$$

Thus

$$\boxed{\nu_t^* = \frac{1}{(T-t) + \frac{\mathbf{a}}{\boldsymbol{\alpha}}}\; Y_t^{\nu^*}\ .}$$

Inventory path

- Inventory path

$$dY_t^{\nu^*} = -\left((T-t) + \tfrac{\mathbf{a}}{\boldsymbol{\alpha}}\right)^{-1} Y_t^{\nu^*}\, dt\,,$$

recall $Y_t^{\nu} = \mathfrak{N} - Q_t^{\nu}$,

$$\boxed{Q_t^{\nu^*} = \frac{t}{T + \frac{\mathbf{a}}{\boldsymbol{\alpha}}}\,\mathfrak{N}\,, \qquad \text{and} \qquad \nu_t^* = \frac{\mathfrak{N}}{T + \frac{\mathbf{a}}{\boldsymbol{\alpha}}}\,.}$$

- For any finite $\boldsymbol{\alpha} > 0$ and finite $\mathbf{a} > 0$,
  - it is always optimal to leave some shares to be executed at the terminal date, and
  - the fraction of shares left to execute at the end decreases with the relative price impact at the terminal date, $a/\boldsymbol{\alpha}$.

Optimal Liquidation with Temporary and Permanent Price Impact

Liquidation with temporary and permanent impact

- $f(\nu) = \mathbf{a}\,\nu$, and $g(\nu) = \mathbf{b}\,\nu$, so that

$$\hat{S}_t^\nu = S_t^\nu - \left(\tfrac{1}{2}\Delta + \mathbf{a}\,\boldsymbol{\nu}\right) ,$$

and

$$dS^\nu = -\mathbf{b}\,\nu\,\mathbf{dt} + \sigma\,dW .$$

- Agent's performance criterion is

$$H^\nu = \mathbb{E}_{t,x,S,q}\Big[\underbrace{X_T^\nu}_{\text{Terminal Cash}} + \underbrace{Q_T^\nu\,(S_T^\nu - \alpha\,Q_T^\nu)}_{\text{Terminal Execution}} - \underbrace{\phi \int_t^T (Q_u^\nu)^2\,du}_{\text{Inventory Penalty}} \Big] ,$$

and value function

$$H(t,x,S,q) = \sup_{\nu\in\mathcal{A}} H^\nu(t,x,S,q) .$$

- Liquidation speed in feedback form

$$\nu^* = \tfrac{1}{2\,\mathbf{a}} \frac{(S\,\partial_x - \mathbf{b}\,\partial_S - \partial_q)H}{\partial_x H}\,.$$

- Ansatz $H(t,x,S,q) = x + S\,q + h(t,S,q)$, $h(T,S,q) = -\alpha\,q^2$:

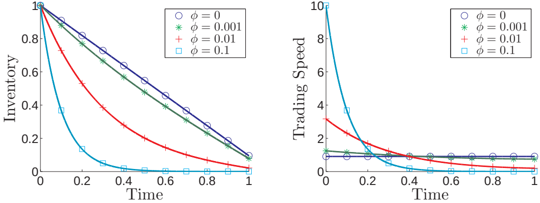
$$0 = \left(\partial_t + \tfrac{1}{2}\sigma^2\,\partial_{SS}\right) h - \phi\,q^2 + \tfrac{1}{4\,\mathbf{a}}\left[\mathbf{b}\,(q + \partial_S h) + \partial_q h\right]^2\,.$$

- After some algebra

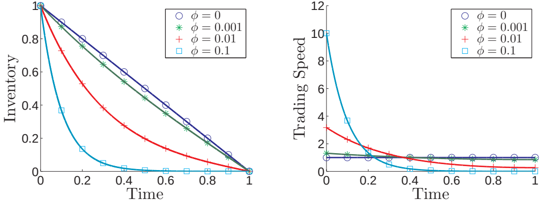
$$\boxed{\nu_t^* = -\sqrt{\frac{\phi}{\mathbf{a}}}\,\frac{1+\zeta\,e^{2\gamma\,(T-t)}}{1-\zeta\,e^{2\gamma(T-t)}}\,Q_t^{\nu^*}\,, \quad Q_t^{\nu^*} = \frac{\zeta\,e^{\gamma(T-t)} - e^{-\gamma(T-t)}}{\zeta\,e^{\gamma T} - e^{-\gamma T}}\,\mathfrak{N}\,,}$$

where

$$\gamma = \sqrt{\frac{\phi}{\mathbf{a}}} \quad \text{and} \quad \zeta = \frac{\alpha - \frac{1}{2}\mathbf{b} + \sqrt{\mathbf{a}\,\phi}}{\alpha - \frac{1}{2}\mathbf{b} - \sqrt{\mathbf{a}\,\phi}}\,.$$



(a) $\alpha = 0.01$



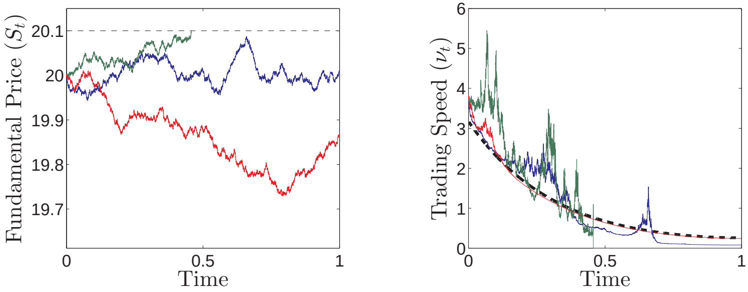
(b) $\alpha = +\infty$

Figure: Other model parameters are **a** $= 10^{-3}$, **b** $= 10^{-3}$.

Acquisition with a price limiter

Optimal Acquisition

If there is a **limit price**, i.e., trader must acquire all shares prior to an upper bound... see Jaimungal & Kinzebulatov (2013)



Now **optimal strategy** is **dependent** on **asset price**

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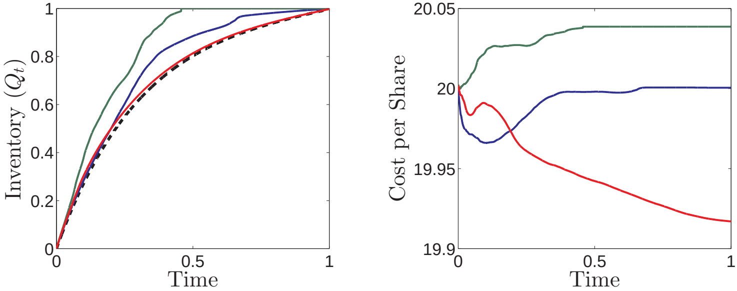

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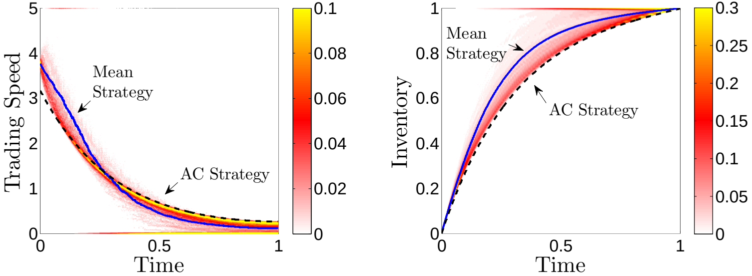

Now **optimal strategy** is **dependent** on **asset price**

Dark Pools

Dark Pools

- **Dark pools** are trading venues which **do not display** bid and ask quotes
- A **crossing network** is a particular kind of dark pool

  > *"...systems that allow participants to enter unpriced orders to buy and sell securities, these orders are crossed at a specified time at a price derived from another market..."*

- The **execution price** is pinned to the **lit market**
- Thus, dark pools have **no temporary impact**, but do have **execution risk**

Dark Pools

Simple Execution Model with Lit and Dark Markets:

- Other **orders arrive** in the dark pool at a **constant rate λ**, let N_t denote the corresponding Poisson process

- Each order has a **random volume** ξ_j, iid

- The agent
  - **trades in the lit market** at a rate of ν_t

  - **places orders** for a volume y_t in the **dark pool**

$$dQ_t^{\nu,y} = -\nu_t\, dt - \min\left(y_t\,,\,\xi_{1+N_{t^-}}\right)\, dN_t\,.$$

- Her **cash process** satisfies the SDE

$$dX_t^{\nu,y} = (S_t - \mathbf{a}\,\nu_t)\,\nu_t\, dt + S_t\,\min\left(y_t\,,\,\xi_{1+N_{t^-}}\right)\, dN_t\,.$$

Dark Pools

- Her **performance criteria** is, as usual, given by

$$H^{\nu,y}(t,x,S,q) = \mathbb{E}_{t,x,S,q}\left[X_\tau^{\nu,y} + Q_\tau^{\nu,y}\,(S_\tau - \alpha\, Q_\tau^{\nu,y}) - \phi \int_t^\tau (Q_u^{\nu,y})^2\, du\right],$$

and

$$\tau = T \wedge \inf\{t \ :\ Q_t = 0\}\,.$$

- The resulting HJB equation is

$$\begin{aligned}
&\partial_t H + \tfrac{1}{2}\sigma^2 \partial_{SS} H - \phi\, q^2 \\
&\qquad + \sup_{\nu}\{(S - \mathbf{a}\,\nu)\,\nu\,\partial_x H - \nu\,\partial_q H\} \\
&\qquad + \sup_{y \le q}\{\lambda\,\mathbb{E}\,[H\,(t, x + S\,\min(y,\xi),\, S,\, q - \min(y,\xi)) - H]\} = 0\,.
\end{aligned}$$

Dark Pools

- Suppose whatever we post in the dark pool is cleared if an order arrives, i.e. $\xi \geq \mathfrak{N}$ a.s.

- In this case, the optimal strategy is

$$\boxed{y_t^* = Q_t^{\nu^*} \qquad \nu_t^* = -\frac{1}{\mathbf{a}}\, h_2(t)\, Q_t^{\nu^*}\,,}$$

  where

$$H = x + q\,S + q^2\,h_2(t)\,.$$

- Moreover, h_2 solves

$$\partial_t h_2 - \boldsymbol{\phi} - \boldsymbol{\lambda}\, h_2 + \tfrac{1}{\mathbf{a}} h_2^2 = 0, \qquad h_2(T) = -\boldsymbol{\alpha}\,.$$

- The above ODE can be solved explicitly

$$h_2(t) = \frac{\zeta^- - \zeta^+\,\beta\,e^{-\gamma(T-t)}}{1-\beta\,e^{-\gamma(T-t)}}\,,$$

$$\zeta^{\pm} = \tfrac{1}{2}\mathbf{a}\,\boldsymbol{\lambda} \pm \sqrt{\tfrac{1}{4}\mathbf{a}^2\,\boldsymbol{\lambda}^2 + \mathbf{a}\boldsymbol{\phi}}\,, \quad \beta = \frac{\boldsymbol{\alpha}+\zeta^-}{\boldsymbol{\alpha}+\zeta^+} \quad \text{and} \quad \gamma = \tfrac{1}{\mathbf{a}}(\zeta^+ - \zeta^-)\,.$$

Dark Pools

- More interesting to look at the infinite terminal penalty case...

$$Q_t^{\nu^*,y^*} \xrightarrow{\alpha\to\infty} \underbrace{e^{\left(\frac{\zeta^-}{a}+\frac{\gamma}{2}\right)t}}_{\text{Dark Pool Adjustment}} \times \underbrace{\frac{\sinh\left(\frac{\gamma}{2}(T-t)\right)}{\sinh\left(\frac{\gamma}{2}T\right)}\mathfrak{N}}_{\text{Almgren-Chriss}}\,.$$