# Mean Field Games with Latent Alpha

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joint work with

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# Introduction

# Introduction

- **Algorithmic and High-Frequency Trading** covers a diverse set of topics, but largely fall into three categories:

  1. **Optimal Execution** – selling/acquiring a large number of assets
  2. **Statistical Arbitrage** – making profits of off predictable returns, i.e., short-term "alpha"
  3. "**Market Making**" – setting bid-ask spreads and/or optimal placement of limit orders

# Introduction

Table: A non-exhaustive list of contributors to this research agenda:

| | | | |
|---|---|---|---|
| Gan | Rosenbaum | Leung | Almgren |
| Obizhaeva | Bouchard | Forsyth | Kinzebulatov |
| Capponi | Stoikov | Shreve | Kukanov |
| Sophie | Bouchaud | Alfonsi | Ricci |
| Elliot | Tourin | Macrina | Muhle-Karbe |
| Moallemi | Lehalle | Horst | Carmona |
| Wang | Gao | Avellaneda | Pham |
| Gatheral | Ludkovski | Scheid | Viens |
| Frey | Westray | Chriss | Nadtochiy |
| Bayraktar | Gueant | Cont | Abergel |
| Bank | Frei | Rogers | You Too |

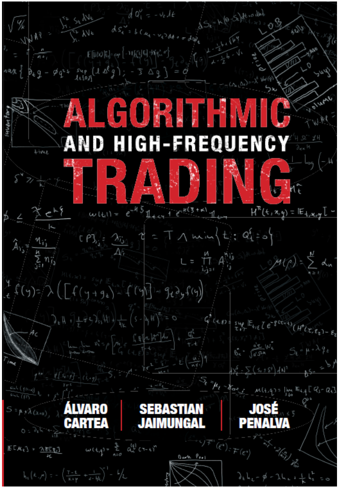
# Introduction

This talk looks at three (related) topics:

- How **order-flow** affects prices, and how to optimally trade with this information
- How **latent alpha** factors can be used to generate statistical arbitrage strategies
- How to include large populations of **heterogenous traders**

from a stochastic control/game perspective.

# Introduction

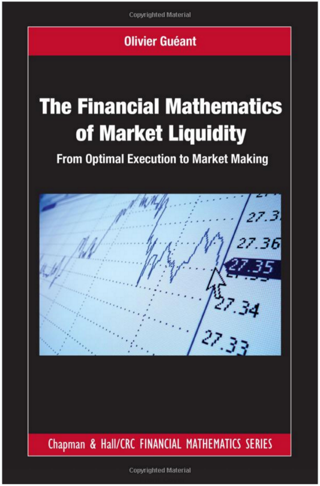


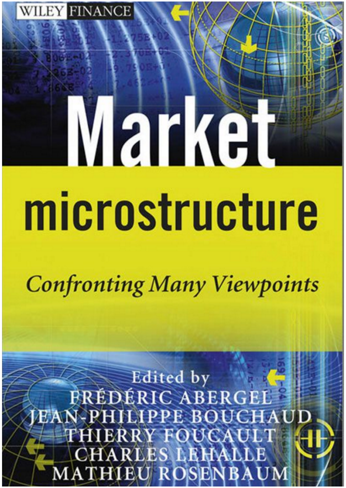

Álvaro Cartea,
Sebastian Jaimungal,
and
José Penalva

Cambridge University Press

(Aug, 2015)

# Introduction



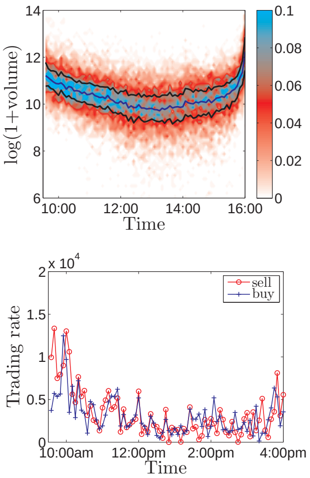
# Order-Flow

# Order-Flow

- **Trade volume** and **intensity** are **stochastic**
- **All order-flow affects prices**... not just "me"
- How can we **use order-flow information** and prediction **to improve trading performance**?

# Order-Flow

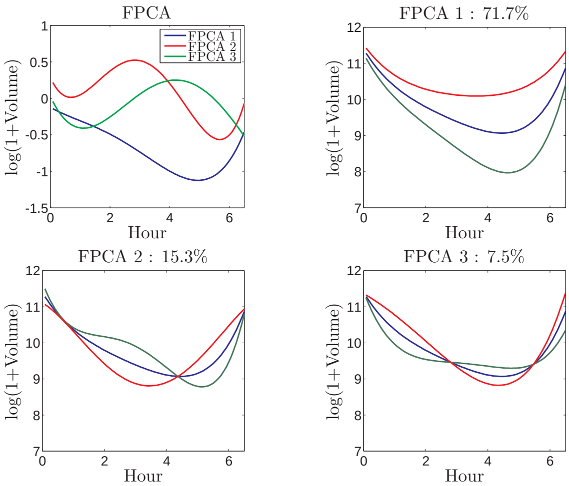
Figure: ORCL (2013) traded volume using 5 minute buckets.




How can we extract some more information?

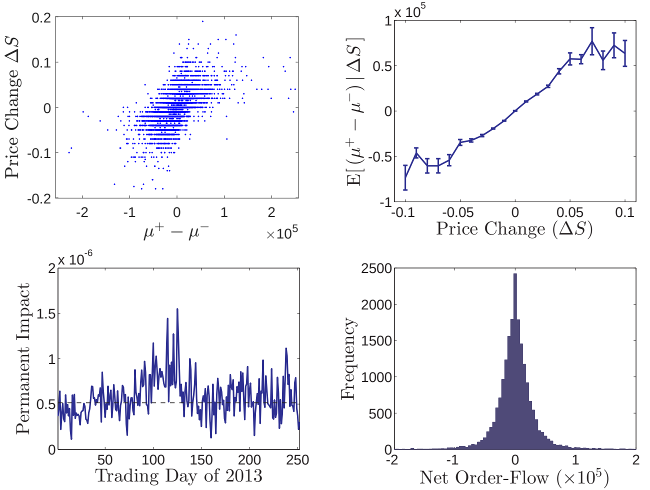
# Order-Flow: FPCA

Figure: First three order-flow **functional principal components** : INTC (2014)



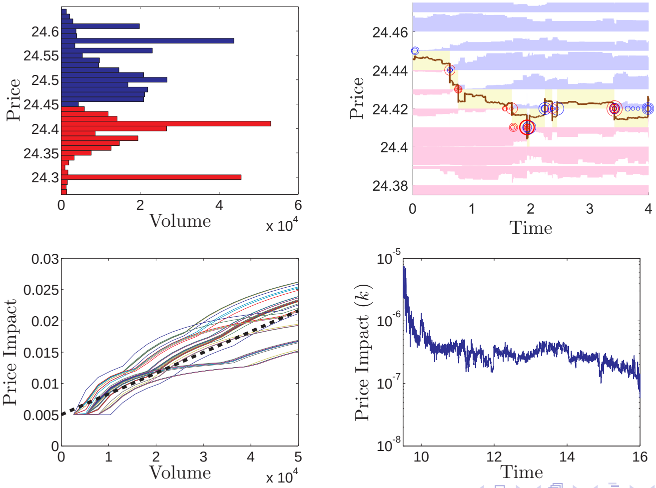
# Order-Flow: Permanent Price Impact

Figure: **Order-Flow** and effect on the **midprice**. INTC (2014)



# Order-Flow: Temporary Price Impact

Figure: The **immediate impact** from **walking the limit order book (LOB)**. INTC on Nov 1, 2013.



# Order-Flow: Mathematical Model

- **Temporary price impact**

$$\hat{S}_t^\nu = S_t^\nu + a\,\boldsymbol{\nu_t}$$

- **Price process** is affected by order-flow from all agents, i.e., **permanent price impact**

$$dS_t^\nu = b\,\underbrace{(\boldsymbol{\nu_t} + \boldsymbol{\mu_t^+} - \boldsymbol{\mu_t^-})}_{\textbf{net order-flow}}\,dt + dM_t$$

- **Cash process** is

$$X_t^\nu = -\int_0^t (S_u^\nu + a\,\nu_u)\,\nu_u\,du$$

# Order-Flow: Mathematical Model

- Agent aims to solve for

$$H(t,x,q,S,\boldsymbol{\mu}) = \sup_{\nu\in\mathcal{A}} \mathbb{E}_{t,x,q,S,\boldsymbol{\mu}}\Bigg[ \overbrace{X_T^\nu + S_T^\nu\, q_T^\nu}^{\text{terminal book-value}} \quad - \quad \overbrace{\alpha\,(q_T^\nu)^2}^{\text{terminal inventory penalty}} \\ - \underbrace{\phi\,\sigma^2 \int_t^T (q_u^\nu)^2\, du}_{\text{running inventory penalty}} \Bigg]$$

- The **running penalty** can be understood as arising from **ambiguity aversion** (model uncertainty) [Cartea, Jaimungal, Donnelly (2015)]

# Order-Flow: Mathematical Model

- If the agent is ambiguity averse, they aim to solve

$$H(t,x,q,S,\boldsymbol{\mu}) = \sup_{\nu\in\mathcal{A}} \inf_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}_{t,x,q,S,\boldsymbol{\mu}}\left[ \overbrace{X_T^\nu + S_T^\nu\, q_T^\nu}^{\text{terminal book-value}} - \overbrace{\boldsymbol{\alpha\,(q_T^\nu)^2}}^{\text{terminal inventory penalty}} + \frac{1}{\varphi}\underbrace{\log \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_t^T}_{\text{ambiguity aversion}} \right]$$

- **Alternate models** have an arbitrary (Markov) drift adjustment in the midprice
- **Model uncertainty** causes the agent to trade as if

$$dS_t^{\nu,\eta^*} = \left[ b\,\underbrace{(\boldsymbol{\nu_t} + \boldsymbol{\mu_t^+} - \boldsymbol{\mu_t^-})}_{\text{net order-flow}} - \overbrace{\varphi\,\boldsymbol{Q_t^{\nu,\eta^*}}}^{\text{ambiguity adjustment}} \right] dt + dM_t$$

# Order-Flow: Optimal Trading

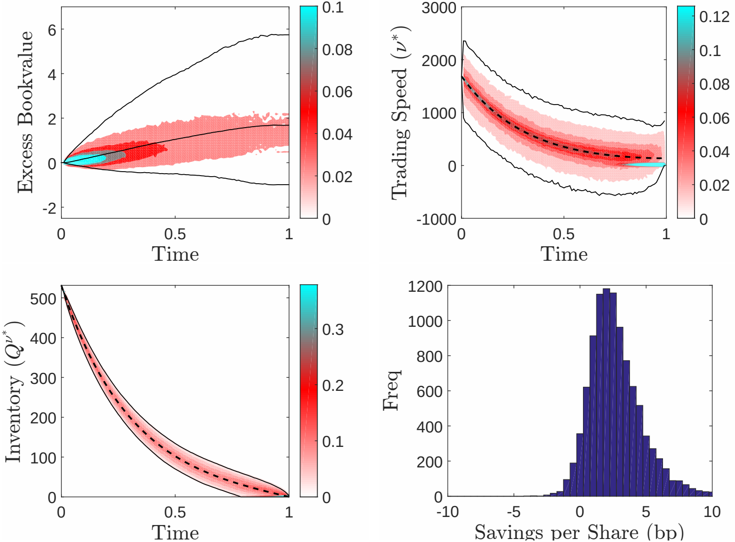
## Theorem

*The optimal trading speed to the control problem is*

$$\lim_{\alpha\to\infty} \nu_t^* = \overbrace{-\gamma \tanh(\gamma\,(T-t))\, \boldsymbol{Q}_t^{\nu^*}}^{\textit{Implementation Shortfall}} + \underbrace{\frac{b}{2k} \int_t^T \frac{\sinh(\gamma(T-u))}{\sinh(\gamma(T-t))}\, \mathbb{E}\left[\, \boldsymbol{\mu}_u^+ - \boldsymbol{\mu}_u^- \,|\boldsymbol{\mathcal{F}}_t^{\boldsymbol{\mu}}\right] du}_{\textit{Order-flow adjustment}}\,.$$

*is the admissible optimal control we seek, and $\mathcal{F}_t^{\boldsymbol{\mu}}$ denotes the natural filtration generated by $\boldsymbol{\mu}$.*

# Order-flow: Simulation Study



# Latent Alpha Models

# Latent Alpha Models

- With **latent alpha signals**, the unimpacted price $F$ satisfies

$$dF_t = \boldsymbol{A_t}\, dt + b\,(d\boldsymbol{N_t^+} - d\boldsymbol{N_t^-}) + \sigma\, dW_t$$

and the midprice with impact is

$$S_t = F_t + \beta \int_0^t \boldsymbol{\nu_u}\, du$$

where $\boldsymbol{\nu}$ is the agent's **speed of trading**

- $\boldsymbol{\Theta_t} \in \{\theta_j\}_{j=1\ldots N}$ is a **hidden Markov chain** that modualtes $\boldsymbol{A_t} = A_t^{\Theta_t}$

- $\boldsymbol{N_t^\pm}$ are **counting processes** with respective **unobserved intensity** processes $\lambda_t^\pm$ of the form

$$\lambda_t^\pm = \sum_{j=1}^N \mathbb{1}_{\{\boldsymbol{\Theta_t}=\theta_j\}}\, \lambda_t^{\pm,j}\,,$$

$\lambda_t^{\pm,j}$ are adapted to the filtration generated by $F$ (denoted $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$) and $(\Theta_t, \lambda_t^{\pm,j}, F_t, N_t^\pm)_{t\geq 0}$ is a Markov process.

# Latent Alpha Models

- The agent's **prior** $\pi_0^j := \mathbb{P}(\Theta_0 = \theta_j)$ on the initial latent state
- The agent aims to solve

$$\sup_{\nu\in\mathcal{A}} \mathbb{E}^{\mathbb{P}}\left[ \overbrace{X_T^\nu + S_T^\nu\, q_T^\nu}^{\text{terminal book-value}} - \underbrace{\alpha\,(q_T^\nu)^2}_{\text{terminal inventory penalty}} - \overbrace{\phi \int_t^T (q_u^\nu)^2\, du}^{\text{running inventory penalty}} \right]$$

where the set of admissible strategies $\mathcal{A}$ are adapted to the **partial information** $\mathcal{F}_t$ rather than the full information $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{F}_t^\Theta$

# Latent Alpha Models: Filtering

- To solve the latent (partial information) control problem, the **best estimate** of $\Theta_t$ given $\mathcal{F}_t$ is required

$$\pi_t^j := \mathbb{E}^{\mathbb{P}}\left[ \mathbb{1}_{\{\Theta_t=\theta_j\}} \middle| \mathcal{F}_t \right]$$

- Key trick: introduce a **measure** $\mathbb{Q}$ s.t. $\boldsymbol{\sigma^{-1}(F_t - bN_t)}$ is a $\mathbb{Q}$-**Brownian motion** and $\boldsymbol{N_t^{\pm}}$ each have **intensity 1**

# Latent Alpha Models: Filtering

## Theorem

*The filter* $\{\pi_t^j\}_{j=1}^M$ *admits the representation*

$$\pi_t^j = \Lambda_t^j \Bigg/ \sum_{i=1}^{M} \Lambda_t^i \quad ,$$

*where* $\{\Lambda_t^j\}_{j=1}^M$ *solve the coupled system of SDEs*

$$\frac{d\Lambda_t^j}{\Lambda_t^j} = \sigma^{-2} A_t^j \,\left(dF_t - b(dN_t^+ - dN_t^-)\right)$$

$$+ (\lambda_{t^-}^{+,j} - 1)(dN_t^+ - dt) + (\lambda_{t^-}^{-,j} - 1)(dN_t^- - dt) + \sum_{i=1}^{N} \frac{\Lambda_t^i}{\Lambda_t^j} C_{j,i}\, dt \,,$$

*with* $\Lambda_0^j = \pi_0^j$

# Latent Alpha Models: Partial to Full Information

## Theorem

*Define the processes* $\hat{W}$, $\hat{M}^{\pm}$ *as*

$$\hat{W}_t = W_t + \sigma^{-1} \int_0^t \left(A_u - \hat{A}_u\right)\, du\,,$$

$$\hat{M}^{\pm}_t = M^{\pm}_t + \int_0^t \left(\lambda^{\pm}_u - \hat{\lambda}^{\pm}_u\right)\, du$$

*where*

$$\hat{A}_t = \sum_{j=1}^M \pi^j(\boldsymbol{\Lambda}_t)\, A^j_t \quad \textit{and} \quad \hat{\lambda}^{\pm}_t = \sum_{j=1}^M \pi^j(\boldsymbol{\Lambda}_t)\, \lambda^{\pm,j}_t\,.$$

(A) *The process* $\hat{W}$ *is an* $\mathcal{F}$*-adapted* $\mathbb{P}$*-Brownian Motion*

(B) *The processes* $\hat{M}^{\pm}$ *are* $\mathcal{F}$*-adapted* $\mathbb{P}$*-martingales*

(C) $[\hat{W}, \hat{M}^{\pm}]_t = 0$ *and* $[\hat{M}^{+}, \hat{M}^{-}]_t = 0$, $\mathbb{P}$*-almost surely*

(D) $N^{\pm}$ *are* $\mathcal{F}$*-adapted doubly stochastic Poisson processes with* $\mathbb{P}$*-intensities* $\hat{\lambda}^{\pm}$

# Latent Alpha Models: Optimal Trading Strategy

The resulting optimal control when the trader must end the day flat is

$$\boxed{\begin{aligned}&\lim_{\alpha\to\infty}\nu_t^\star\\&\quad=\overbrace{-\gamma\tanh\left(\gamma(T-t)\right)\;\boldsymbol{Q_t^{\nu^\star}}}^{\textbf{Implementation Shortfall}}\\&\qquad\underbrace{+\frac{1}{2a}\int_t^T\mathbb{E}\left[\boldsymbol{\hat{A}_u+b\left(\hat{\lambda}_u^+-\hat{\lambda}_u^-\right)}\Big|\,\mathcal{F}_t\right]\times\frac{\sinh\left(\gamma(T-u)\right)}{\sinh\left(\gamma(T-t)\right)}du}_{\textbf{latent alpha correction}}\;.\end{aligned}}$$

# Latent Alpha Models: Numerical Example

- Let's consider the following mean-reverting dynamics for the asset midprice,

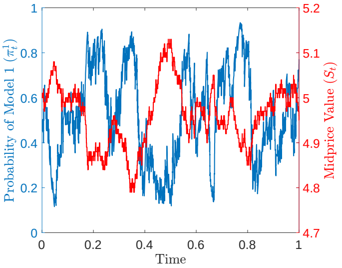
$$dS_t = b\left(dN_t^+ - dN_t^-\right),$$

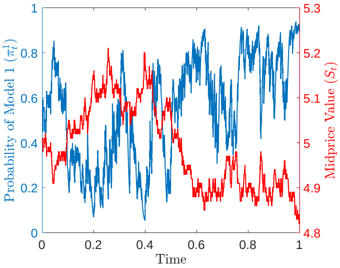
where $N_t^{\pm}$ are doubly stochastic Poisson processes with respective intensities

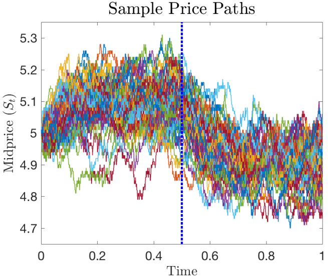
$$\lambda_t^+ = a + \kappa\,(\Theta_t - S_t)_+$$
$$\lambda_t^- = a + \kappa\,(\Theta_t - S_t)_-$$

- $\Theta_t$ is a Markov chain
- $\pi_t^k$ can estimated in an **online** fashion

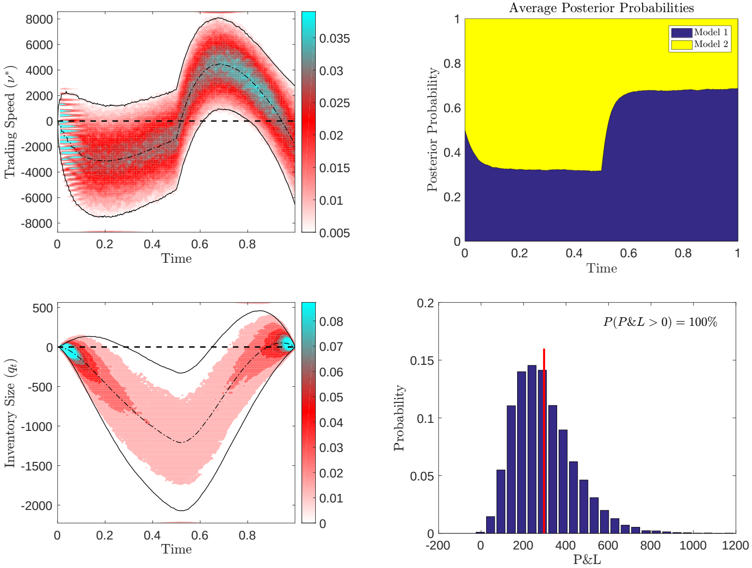
# Latent Alpha Models: Numerical Example



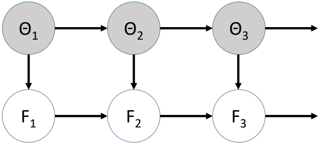
# Latent Alpha Models: Numerical Example



# Calibration

# Calibration

- Parameters need to be learned offline
- **Graphical model** framework with a **hidden layer**




- **completed-data log-likelihood** is

$$
\begin{aligned}
\log L^{\Gamma} = &\sum_{d=1}^{D}\sum_{i=1}^{M} \log\left(\pi_0^i\right)\, \mathbb{1}_{\{\Theta_0^d=\theta_i\}} \\
&+ \sum_{d=1}^{D}\sum_{k=0}^{K-2}\sum_{i,j=1}^{M} \log\left(P_{i,j}\right)\, \mathbb{1}_{\{\Theta_k^d=\theta_i,\, \Theta_{k+1}^d=\theta_j\}} \\
&+ \sum_{d=1}^{D}\sum_{k=0}^{K-1}\sum_{i=1}^{M} \log\left(g_\psi(t_{k+1}, f_{k+1}^d; t_k, f_k^d, \theta_i)\right)\, \mathbb{1}_{\{\Theta_k^d=\theta_i\}} .
\end{aligned}
$$

# Calibration

In the **m-step** find

$$\Gamma_{n+1} = \arg\max_{\Gamma} \mathbb{E}^{\mathbb{P}^{\Gamma_n}} \left[\log L^{\Gamma}\right] = \sum_{d=1}^{D} \sum_{i=1}^{M} \log\left(\pi_0^i\right) \gamma_0^{i,d} + \sum_{d=1}^{D} \sum_{k=0}^{K-2} \sum_{i,j=1}^{M} \log\left(P_{i,j}\right) \xi_k^{i,j,d} + \sum_{d=1}^{D} \sum_{k=0}^{K-1} \sum_{i=1}^{M} \log\left(g_\psi(t_{k+1}, f_{k+1}^d; t_k, f_k^d, \theta_i)\right) \gamma_k^{i,d} ,$$

where the **smoother** $\gamma_k^{i,d}$ and **two-slice marginal** $\xi_k^{i,j,d}$ are defined as

$$\gamma_k^{i,d} = \mathbb{P}^{\Gamma_n}\left(\Theta_{k,i}^d = \theta_i \mid F_{0:K}^d = f_{0:K}^d\right) \quad \text{and}$$

$$\xi_k^{i,j,d} = \mathbb{P}^{\Gamma_n}\left(\Theta_{k,i}^d = \theta_i, \Theta_{k+1,j}^d = \theta_j \mid F_{0:K}^d = f_{0:K}^d\right) .$$

and are computed in the **e-step**.

# Calibration

- The smoother & two-slice marginal can represented via
  - **forward message passing**

$$\alpha_k^j = \mathbb{P}(\Theta_t = \theta^j \mid |F_{1:t} = f_{1:t})$$
$$\eta_k = \mathbb{P}(F_t \mid F_{1:t-1} = f_{1:t-1})$$

  - **backward message passing**

$$\beta_k^j = \frac{\mathbb{P}(F_{k+1:T} = f_{k+1:T} \mid \Theta_k = \theta^j)}{\mathbb{P}(F_{k+1:K} = f_{k+1:K} \mid F_{1:k} = f_{1:k})}$$

- so that

$$\gamma_k^j = \alpha_k^j \, \beta_k^j \qquad \text{and} \qquad \xi_k^{i,j} = \frac{\alpha_k^i \, \beta_{k+1}^j}{c_{k+1}^d} \, P_{i,j} \, f_\psi(t_{k+1}, y_{k+1}^d; t_k, \theta_i, y_k^d)$$

# Calibration

Moreover, $\alpha$ and $\beta$ can be computed **recursively**

- $$\alpha_k^{j,d} = \hat{\alpha}_k^{j,d} \Bigg/ \sum_{j=1}^{M} \hat{\alpha}_k^{j,d} \quad ,$$

  where

  $$\hat{\alpha}_k^{j,d} = \sum_{i=1}^{M} P_{i,j} \, g_\psi(t_k \, f_k^d; t_{k-1}, \theta_i, f_{k-1}^d) \, \alpha_{k-1}^{i,d} \, ,$$

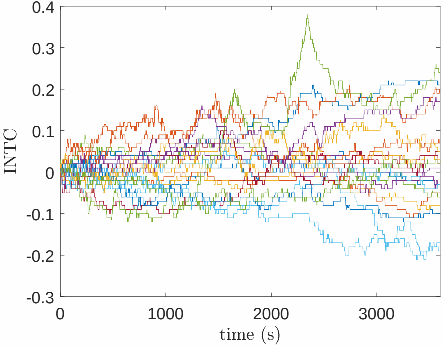
- $$\beta_k^{j,d} = \frac{\hat{\beta}_k^{j,d}}{\sum_{i=1}^{M} \hat{\beta}_k^{i,d} \alpha_n^{i,d}} \, ,$$

  where

  $$\hat{\beta}_k^{j,d} = g_\psi(t_{k+1}, f_{k+1}^d; t_k, \theta_j, f_k^d) \sum_{i=1}^{M} \beta_{k+1}^{i,d} P_{j,i} \; .$$

# Calibration

Figure: **INTC** all of Jan, 2014 – 10am-11am



# Calibration

Table: **INTC** all of Jan, 2014 – **2 Latent States**

| | | $A_{ij}$ | |
|---|---|---|---|
| $i$ | $\pi_i$ | $j=1$ | $j=2$ |
| 1 | 0.000 | 0.995 | 0.005 |
| 2 | 1.000 | 0.013 | 0.987 |
| | $\mu_i$ | $\kappa_i$ | $\theta_i$ |
| 1 | 0.011 | 0.024 | 2.006 |
| 2 | 0.053 | 0.046 | -0.719 |

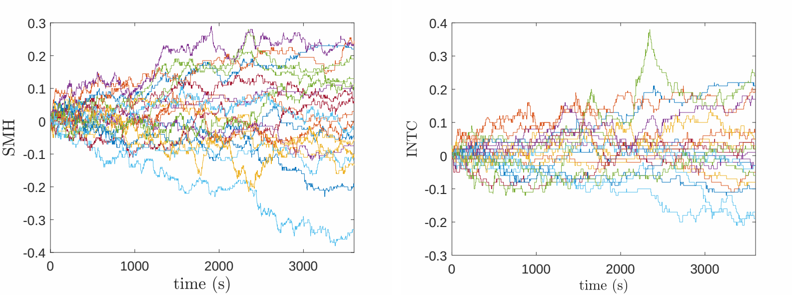
# Calibration

Table: **INTC** all of Jan, 2014 – **3 Latent States**

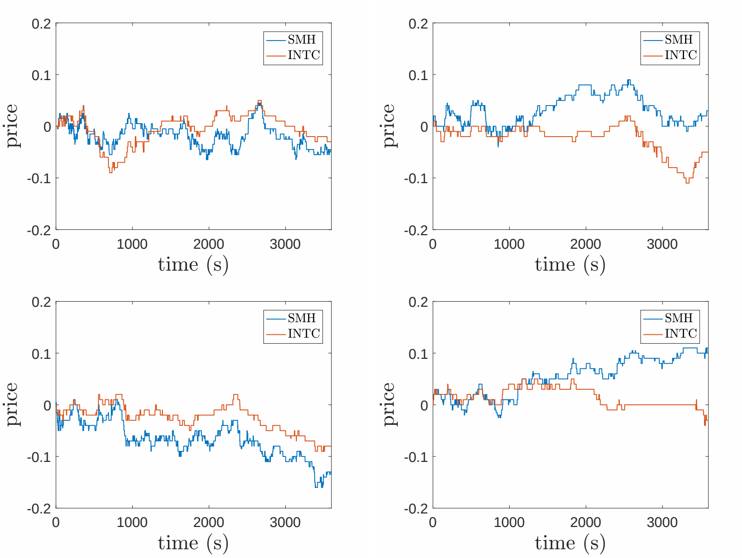
| | | $A_{ij}$ | | |
|---|---|---|---|---|
| $i$ | $\pi_i$ | $j=1$ | $j=1$ | $j=3$ |
| 1 | 0.000 | 0.999 | 0.001 | 0.000 |
| 2 | 0.001 | 0.000 | 0.988 | 0.012 |
| 3 | 0.999 | 0.002 | 0.028 | 0.970 |
| | $\mu_i$ | $\kappa_i$ | $\theta_i$ | |
| 1 | 0.010 | 0.025 | 1.147 | |
| 2 | 0.019 | 0.008 | 3.497 | |
| 3 | 0.070 | 0.071 | -0.485 | |

# Calibration

Figure: **INTC** & **SMH** all of Jan, 2014



# Latent Alpha Models: Numerical Example



# Calibration

Table: Calibration of INTC – SMH 1st month of 2014

| | | | $A_{ij}$ | | |
|---|---|---|---|---|---|
| | | $\pi_i$ | $j = 1$ | $j = 1$ | |
| | $i = 1$ | 0.000 | 0.918 | 0.082 | |
| | $i = 2$ | 1.000 | 0.204 | 0.796 | |
| | | $\mu_i$ | $\kappa_i$ | | $\theta_i$ |
| $i = 1$ | INTC | 0.010 | 0.000 | 0.000 | 0.052 |
| | SMH | 0.024 | 0.007 | -0.006 | 7.894 |
| $i = 2$ | INTC | 0.063 | 0.100 | -0.023 | -0.334 |
| | SMH | 0.299 | -0.088 | 0.045 | -2.029 |

# Mean Field Games

# MFGs

- **Agents' trading rates** $\boldsymbol{\nu} = (\nu_t^1, \dots, \nu_t^N)_{t\in[0,T]} \in \mathcal{A}$
- **Agent's strategies are $\mathcal{F}$-adapted**, where $\mathcal{F} = \sigma((F_u)_{u\le t})$

$$F_t = S_t^{\nu} - \tfrac{\lambda}{N} \textstyle\sum_{i=1}^N q_t^i$$

  The midprice subtract total "order-flow".

- Asset **midprice** satisfies the SDE

$$dS_t^{\nu} = \left( \boldsymbol{A_t} + \tfrac{\lambda}{N} \textstyle\sum_{i=1}^N \nu_t^i \right) dt + dM_t$$

  $A$ **is $\mathcal{G}$-predictable** – latent
  $M$ is a $\mathcal{G}$-martingale.

# MFGs

**Lemma ($\mathcal{F}$-projection.)**

*There exists an $\mathcal{F}$-adapted square-integrable martingale $\hat{M}$ s.t.*

$$dF_t = \hat{A}_t \, dt + d\hat{M}_t$$

*where $\hat{A}_t \triangleq \mathbb{E}[A_t | \mathcal{F}_t]$.*

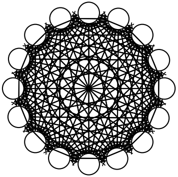
# MFGs

- $i^{th}$-agent's performance criteria is

$$J(\nu^i;\nu^{-i}) = \mathbb{E}\left[\int_0^T \left\{ \left(\boldsymbol{S}_u^{\boldsymbol{\nu}} - a\,\boldsymbol{\nu}_u^i\right)\boldsymbol{\nu}_u^i - \phi\left(q_u^{\nu^i}\right)^2 \right\} du + q_T^{\nu^i}\left(\boldsymbol{S}_T^{\boldsymbol{\nu}} - \gamma\, q_T^{\nu^i}\right)\right]$$

price is impacted by ALL traders, and contains latent risk factors, but own rate affects own inventory.

- Difficult to solve for finite number of players, so look for a MFG solution and apply to the finite player



- Assume a mean-field $\bar{\nu}$
- Find $i^{th}$ agent's optimal strategy
- Average over agents
- Solve the fixed-point problem on space of controls

# MFGs

- suppose $\bar{\nu} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \nu^i$ is given and $\mathcal{F}$-predictable
- We use the convex analysis approach to solving the MFG problem

**Lemma (Gâteaux Differentiable.)**
*The performance criteria $J$ is everwhere Gâteaux differentiable in $\mathcal{A}$. Moreover, given $\nu, \omega \in \mathcal{A}$, the Gâteaux derivative of $J$ in the direction $\omega$ at $\nu$ is*

$$\langle J'(\nu), \omega \rangle = \mathbb{E}\left[ \int_0^T \omega_t \, \mathbb{E}\left[ -2\, a\, \nu_t - 2\gamma q_T^\nu + \int_t^T \left\{ \hat{A}_u + \lambda\, \bar{\nu}_u - 2\,\phi\, q_u^\nu \right\} \middle| \mathcal{F}_t \right] dt \right]$$

# MFGs

### Proposition (Necessary & Sufficient vanishing Gâteaux Derivative .)

$$\left\langle J'(\nu), \omega \right\rangle = 0 \quad \forall \omega \in \mathcal{A}$$

*if and only if $\nu$ is the unique strong solution to the FBSDE*

$$\begin{cases} -d(2\,\kappa\,\nu_t) = \left(\hat{A}_t + \lambda\,\bar{\nu}_t - 2\,\phi\,q_t^\nu\right)\,dt - d\mathcal{M}_t \\ \quad 2\,\kappa\,\nu_T = -\gamma\,q_T^\nu \end{cases} \tag{1}$$

*for a suitable square-integrable martingale $\mathcal{M}_t$.*

### Corollary (The solution to (1) is optimal)

*The solution $\nu$ to (1) is the optimal trading strategy for the $i^{th}$ agent, i.e.,*

$$\nu = \sup_{\chi \in \mathcal{A}} J(\chi)$$

# MFGs

- When there are $K$ **sub-populations** with parameters $(a^k, \phi^k, \gamma^k)_{k=1,\dots,K}$ forming $(p_1, \dots, p_K)$ percentage of the total population, the **fixed-point problem** is

$$\begin{cases} -d(2\boldsymbol{a}\bar{\boldsymbol{\nu}}_t) = \left(\mathbf{1}^{(K\times 1)}\hat{A}_t + \lambda \boldsymbol{P}\,\bar{\boldsymbol{\nu}}_t - 2\phi\,\bar{\boldsymbol{q}}_t\right)\,dt - d\boldsymbol{\mathcal{M}}_t \\ 2\boldsymbol{a}\,\bar{\boldsymbol{\nu}}_T = -\gamma\,\bar{\boldsymbol{q}}_T \end{cases},$$

where

$$\boldsymbol{a} = \operatorname{diag}\left(\{a^k\}_{k=1}^K\right), \qquad \phi = \operatorname{diag}\left(\{\phi^k\}_{k=1}^K\right)$$

$$\gamma = \operatorname{diag}\left(\{\gamma^k\}_{k=1}^K\right), \qquad \boldsymbol{P} = \begin{pmatrix} p_1 & \dots & p_K \\ \vdots & & \vdots \\ p_1 & \dots & p_K \end{pmatrix},$$

# MFGs

## Proposition ($K$ sub-population solution)

*The mean field optimal trading rates are (stacked in a vector)*

$$\bar{\boldsymbol{\nu}}_t = \tfrac{1}{2}\boldsymbol{a}^{-1}\left(\boldsymbol{g}_{1,t} + \boldsymbol{g}_{2,t}\bar{\boldsymbol{q}}_t\right)$$

*where*

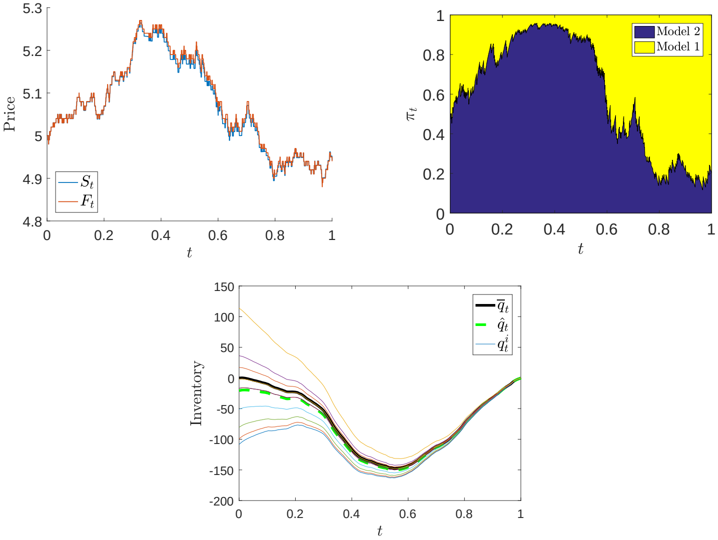
$$\boldsymbol{g}_{1,t} = \int_t^T : e^{\int_u^T \left(\lambda\, \boldsymbol{P} + \boldsymbol{g}_{2,s}\right)(2\,\boldsymbol{\kappa})^{-1}\, ds} : \mathbf{1}^{(K\times 1)}\, \mathbb{E}[\hat{\boldsymbol{A}}_u | \mathcal{F}_t]\, du\,.$$

*and $\boldsymbol{g}_{2,t}$ solves a deterministic matrix ODE, while the $i^{th}$-agent within population $k$–trades according to*

$$\nu_t^{i,k} = \frac{1}{2a^k} h_{2,t}^k (q_t^{i,k} - \bar{q}_t^k) + \bar{\nu}_t^k\,.$$

*and $h_{2,t}^k = -a_k \tanh(b_k + c_k(T - t))$.*

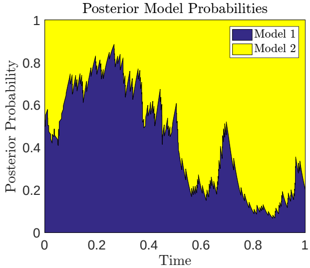
# MFGs



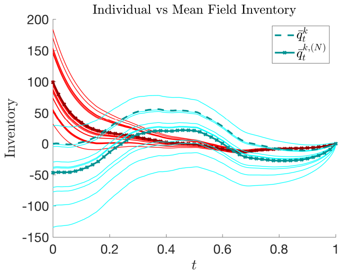
Price
$S_t$
$F_t$
$t$
$\pi_t$
Model 2
Model 1
Inventory
$\overline{q}_t$
$\hat{q}_t$
$q_t^i$

# MFGs

Price Processes and Latent Process
Price
$t$
$S_t$
$F_t$
$\Theta_t$

Posterior Model Probabilities
Posterior Probability
Time
Model 1
Model 2

Individual vs Mean Field Inventory
Inventory
$t$
$\bar{q}_t^k$
$\bar{q}_t^{k,(N)}$

# Conclusions

# Conclusions

- Order-flow and latent-alpha are very similar
- Agent's modify trades based on average future expected latent-alpha
- Heterogeneous agents' modify to account for other agents in a manner that pulls their inventory towards the crowd

# Thank you for your attention!

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